

Exact scaling solution of the mode coupling equations for non-linear fluctuating hydrodynamics in one dimension

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We obtain the exact solution of the one-loop mode-coupling equations for the dynamical structure function in the framework of non-linear fluctuating hydrodynamics in one space dimension for the strictly hyperbolic case where all characteristic velocities are different. All solutions are characterized by dynamical exponents which are Kepler ratios of consecutive Fibonacci numbers, which includes the golden mean as a limiting case. The scaling form of all higher Fibonacci modes are asymmetric Lévy-distributions. Thus a hierarchy of new dynamical universality classes is established. We also compute the precise numerical value of the Prähofer-Spohn scaling constant to which scaling functions obtained from mode coupling theory are sensitive.

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I. INTRODUCTION

Recently new insights into the dynamical universality classes of nonequilibrium systems have been gained. In the presence of slow modes due to locally conserved currents (such as energy, momentum, density etc.) not only the well-established diffusive and the Kardar-Parisi-Zhang (KPZ) universality classes arise in one space dimension, but also a heat mode and other long-lived modes with unexpected scaling properties were discovered [1–3]. Going further we have demonstrated in [4] that in the presence of several conserved quantities there is an infinite family of dynamical universality classes that is characterized by dynamical exponents which take the form of Kepler ratios $z_i = F_{i+2}/F_{i+1}$ where $F_i = 1, 1, 2, 3, 5, \dots$ are the Fibonacci numbers defined recursively by $F_i = F_{i-1} + F_{i-2}$ with starting values $F_1 = F_2 = 1$. This conclusion was based on a scaling analysis of the mode coupling equations for non-linear fluctuating hydrodynamics (NLFH) [5] and supported by extensive Monte-Carlo simulations of multi-lane asymmetric exclusion processes. The first level of the hierarchy (apart from the usual diffusion with $z = 2 = z_1$) includes the Kardar-Parisi-Zhang (KPZ) universality class with $z = 3/2 = z_2$ which continues to inspire both due to its links to intriguing mathematical problems and beautiful experimental results, see e.g. the special issue J. Stat. Phys. 160, (2015) dedicated to it, and in particular the review by Halpin-Healey and Takeuchi [6]. Also the golden mean, which is the limiting Kepler ratio $z_\infty = \varphi \approx 1.618\dots$, occurs in systems with at least two conservation laws [2, 3].

NLFH has emerged as a universal tool to analyze general one-dimensional systems such as Hamiltonian dynamics [1, 7], anharmonic chains [3, 5, 8–11] or driven diffusive systems [2, 4, 12–16]. The theory is robust. The essential ingredients appear to be only the above-mentioned locally conserved currents and long-time dynamics dominated by the long wavelength modes of the associated conserved quantities. Mathematically rigorous results for some specific models support the validity of the theory [17, 18]. It is the purpose of this work to provide a detailed analysis of the one-loop mode-coupling equations for the dynamical structure function for an arbitrary number of conservation laws in the strictly hyperbolic setting where all characteristic velocities are different.

II. COMPUTATION OF THE DYNAMICAL STRUCTURE FUNCTION

A. Basis of nonlinear fluctuating hydrodynamics

Consider an interacting system with n locally conserved currents j_λ associated to physical quantities such as energy, momentum, particle numbers etc. that are conserved under the microscopic dynamics of the system. The starting point for investigating the large-scale dynamics is the system of conservation laws

$$\frac{\partial}{\partial t}\vec{\rho}(x, t) + \frac{\partial}{\partial x}\vec{j}(x, t) = 0 \quad (1)$$

where component $\rho_\lambda(x, t)$ of the vector $\vec{\rho}(x, t)$ is a coarse-grained conserved quantity and the component $j_\lambda(x, t)$ of the current vector $\vec{j}(x, t)$ is the associated locally conserved current. We shall refer to the $\rho_\lambda(x, t)$ as densities. Notice that in our convention $\vec{\rho}$ and \vec{j} are regarded as column vectors. Transposition is denoted by a superscript T .

This system of conservation laws can be obtained from the law of a large numbers and the postulate of local equilibrium [19, 20]. Thus the current is a function of x and t only through its dependence on the local conserved densities. Hence these equations can be rewritten as

$$\frac{\partial}{\partial t}\vec{\rho}(x, t) + \bar{\mathbf{J}}\frac{\partial}{\partial x}\vec{\rho}(x, t) = 0 \quad (2)$$

where $\bar{\mathbf{J}} \equiv \bar{\mathbf{J}}(\vec{\rho}(x, t))$ is the current Jacobian with matrix elements $\bar{J}_{\lambda\mu} = \partial j_\lambda / \partial \rho_\mu$.

To get some basic insight consider we first notice that constant densities ρ_λ are a (trivial) stationary solution of (2). Stationary fluctuations of the conserved quantities are captured in the covariance matrix \mathbf{K} of the conserved quantities that we shall not describe explicitly. However, we have in mind the generic case where \mathbf{K} is positive definite, i.e., we do not allow for vanishing fluctuations of a locally conserved quantity that can occur in systems with slowly decaying stationary correlations. We shall refer to \mathbf{K} as compressibility matrix.

Expanding the local densities $\rho_\lambda(x, t) = \rho_\lambda + u_\lambda(x, t)$ around their long-time stationary values ρ_λ and taking a linear approximation (where $\bar{\mathbf{J}}$ is a constant matrix $\mathbf{J} \equiv \mathbf{J}(\vec{\rho})$ with matrix elements determined by the stationary densities $\vec{\rho}$) leads to a system of coupled linear PDE's which is solved by diagonalizing \mathbf{J} . One transforms to normal modes $\vec{\phi} = \mathbf{R}\vec{u}$ where $\mathbf{R}\mathbf{J}\mathbf{R}^{-1} = \text{diag}(v_\alpha)$ and the transformation matrix \mathbf{R} is normalized such that $\mathbf{R}\mathbf{K}\mathbf{R}^T = \mathbf{1}$. Thus one finds decoupled equations $\partial_t \phi_\alpha = v_\alpha \partial_x \phi_\alpha$ whose solutions are travelling waves

$\phi_\alpha(x, t) = \phi_\alpha^0(x - v_\alpha t)$ with initial data $\phi_\alpha(x, 0) = \phi_\alpha^0(x)$. This shows that the eigenvalues v_α of \mathbf{J} play the role of characteristic speeds.

The product \mathbf{JK} of the Jacobian with the compressibility matrix \mathbf{K} is symmetric which can be proved already on microscopic level [21] for sufficiently fast decaying stationary correlations. This guarantees that on macroscopic scale the full non-linear system (2) is hyperbolic [22], i.e., all eigenvalues v_α of \mathbf{J} are guaranteed to be real. If the eigenvalues v_α are non-degenerate the system is called strictly hyperbolic. The occurrence of complex eigenvalues signals macroscopic phase separation [16], consistent with the absence of fast decaying stationary correlations on microscopic level, and coarsening dynamics.

Notice that (2) is completely deterministic. In the NLFH approach [5] the effect of fluctuations is captured by adding a phenomenological diffusion matrix D and white noise terms ξ_i . This turns (2) into a non-linear stochastic PDE. From renormalization group considerations it is known that polynomial non-linearities of order higher than 4 are irrelevant for the large-scale behaviour and order 3 leads at most to logarithmic corrections if the generic quadratic non-linearity is absent [23]. Thus one expands $\bar{\mathbf{J}}$ around the stationary densities $\vec{\rho}$ but keeps only quadratic non-linearities so that the fluctuation fields $u_\lambda(x, t)$ satisfy the system of coupled noisy Burgers equations

$$\partial_t \vec{u} = -\partial_x \left(J_0 \vec{u} + \frac{1}{2} \vec{u}^T \vec{H} \vec{u} - D \partial_x \vec{u} + B \vec{\xi} \right) \quad (3)$$

where \vec{H} is a column vector whose entries $(\vec{H})_\lambda = \mathbf{H}^\lambda$ are the Hessians with matrix elements $H_{\mu\nu}^\lambda = \partial^2 j_\lambda / (\partial \rho_\mu \partial \rho_\nu)$. If the quadratic non-linearity is absent one has diffusive behaviour.

Using normal modes one thus arrives at

$$\partial_t \phi_\alpha = -\partial_x \left(v_\alpha \phi_\alpha + \vec{\phi}^T G^\alpha \vec{\phi} - \partial_x (\tilde{D} \vec{\phi})_\alpha + (\tilde{B} \vec{\xi})_\alpha \right) \quad (4)$$

with $\tilde{\mathbf{D}} = \mathbf{R} \mathbf{D} \mathbf{R}^{-1}$ and $\tilde{\mathbf{B}} = \mathbf{R} \mathbf{B}$. The matrices

$$\mathbf{G}^\alpha = \frac{1}{2} \sum_\lambda R_{\alpha\lambda} (\mathbf{R}^{-1})^T \mathbf{H}^\lambda \mathbf{R}^{-1} \quad (5)$$

are the mode coupling matrices with the mode-coupling coefficients $G_{\beta\gamma}^\alpha = G_{\gamma\beta}^\alpha$ which are, by construction, symmetric. From the linear theory one concludes that the fluctuation fields are peaked around $x_\alpha(t) = x_\alpha(0) + v_\alpha t$. For short-range interactions fluctuations spread generally sub-ballistically and therefore the width of the peak grows in sublinearly time, as indeed will be seen explicitly below.

We stress that the macroscopic current-density relation given by the components of the current vector \vec{j} arises from the microscopic model from the stationary current-density relation $\vec{j}(\vec{\rho})$. Similarly, the compressibility matrix \mathbf{K} is computed from the stationary distribution of the microscopic model. Hence the mode coupling matrices (and with them the dynamical universality classes as shown below) are completely determined by these two macroscopic stationary properties of the system. However, the *exact* stationary current-density relations and the *exact* stationary compressibilities are required.

The main quantity of interest are the dynamical structure functions

$$S^{\alpha\beta}(x, t) = \langle \phi^\alpha(x, t) \phi^\beta(0, 0) \rangle \quad (6)$$

(where $\langle \dots \rangle$ denotes the stationary ensemble average) which describe the stationary space-time fluctuations. Since we work with normal modes we have the normalization

$$\int_{-\infty}^{\infty} dx S^{\alpha\beta}(x, t) = \delta_{\alpha,\beta}. \quad (7)$$

For strictly hyperbolic systems the characteristic velocities are all different. As a result the off-diagonal elements of S decay quickly and for long times and large distances one is left with the diagonal elements which we denote by

$$S_\alpha(x, t) := S^{\alpha\alpha}(x, t). \quad (8)$$

The large scale behaviour of the diagonal elements is expected to have the scaling form

$$S_\alpha(x, t) \sim t^{-1/z_\alpha} f_\alpha(\xi_\alpha) \quad (9)$$

with the scaling variable

$$\xi_\alpha = (x - v_\alpha t) t^{-1/z_\alpha} \quad (10)$$

and dynamical exponent z_α which has to be determined and which indicates the dynamical universality class of the mode α . The exponent in the power law prefactor follows from the conservation law. In momentum space, with the Fourier transform convention

$$\hat{S}_\alpha(k, t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-ikx} S_\alpha(x, t) \quad (11)$$

one has the scaling form

$$\hat{S}_\alpha(k, t) \sim e^{-iv_\alpha kt} \hat{f}_\alpha(kt^{1/z_\alpha}) \quad (12)$$

where \hat{f}_α is the Fourier transform of the scaling function (9).

B. Mode coupling equations

The starting point for computing the diagonal elements of the dynamical structure function are the mode coupling equations [5]

$$\partial_t S_\alpha(x, t) = D_\alpha S_\alpha(x, t) + \int_0^t ds \int_{-\infty}^{\infty} dy S_\alpha(x - y, t - s) M_{\alpha\alpha}(y, s) \quad (13)$$

with diffusion operator

$$D_\alpha = -v_\alpha \partial_x + D_\alpha \partial_x^2 \quad (14)$$

and memory term

$$M_{\alpha\alpha}(y, s) = 2 \sum_{\beta, \gamma} (G_{\beta\gamma}^\alpha)^2 \partial_y^2 S_\beta(y, s) S_\gamma(y, s). \quad (15)$$

Only the diagonal elements $D_\alpha := D_{\alpha\alpha}$ of the diffusion matrix and of the memory kernel are kept here.

In momentum space this reads

$$\partial_t \hat{S}_\alpha(k, t) = -\hat{D}_\alpha(k) \hat{S}_\alpha(k, t) - \int_0^t ds \hat{S}_\alpha(k, t - s) \widehat{M}_{\alpha\alpha}(k, s) \quad (16)$$

with

$$\hat{D}_\alpha(k) = i v_\alpha k + D_\alpha k^2 \quad (17)$$

and

$$\widehat{M}_{\alpha\alpha}(k, s) = 2 \sum_{\beta, \gamma} (G_{\beta\gamma}^\alpha)^2 k^2 \int_{-\infty}^{\infty} dq \hat{S}_\beta(q, s) \hat{S}_\gamma(k - q, s). \quad (18)$$

Finally we perform the Laplace transformation

$$\tilde{S}_\alpha(k, \omega) := \int_0^{\infty} dt e^{-\omega t} \hat{S}_\alpha(k, t) \quad (19)$$

by multiplying (16) on both sides by $e^{-\omega t}$ and integrating over t . This yields

$$\tilde{S}_\alpha(k, \omega) = \frac{\hat{S}_\alpha(k, 0)}{\omega + \hat{D}_\alpha(k) + \tilde{C}_{\alpha\alpha}(k, \omega)} \quad (20)$$

with memory kernel

$$\tilde{C}_{\alpha\alpha}(k, \omega) = 2 \sum_{\beta, \gamma} (G_{\beta\gamma}^\alpha)^2 k^2 \int_0^{\infty} ds e^{-\omega s} \int_{-\infty}^{\infty} dq \hat{S}_\beta(q, s) \hat{S}_\gamma(k - q, s). \quad (21)$$

and $\hat{S}_\alpha(k, 0) = 1/\sqrt{2\pi}$.

Remark II.1 For $k = 0$ the solution is trivial, with the exact result $\hat{S}_\alpha(0, t) = 1/\sqrt{2\pi}$ given by the Fourier convention (11) and the normalization (7).

So far this is an exact reformulation of the original mode coupling equations (13). In order to proceed we make impose successively various conditions (Conditions 1 - 3). We stress that conditions 1 and 2 do not lead to any loss of generality in the subsequent treatment.

Condition 1: Scaling ($k \neq 0$).

The mode coupling equation (16) can be further analyzed using the scaling form (12). To this end we first rewrite (20) in terms of $\tilde{\omega}_\alpha := \omega + iv_\alpha k$. This yields

$$\tilde{S}_\alpha(k, \tilde{\omega}_\alpha) = \hat{S}_\alpha(k, 0) \left[\tilde{\omega}_\alpha + D_\alpha k^2 + 2 \sum_{\beta, \gamma} (G_{\beta\gamma}^\alpha)^2 I_{\beta\gamma}(k, \tilde{\omega}_\alpha) \right]^{-1} \quad (22)$$

with modified memory integral

$$I_{\beta\gamma}(k, \tilde{\omega}_\alpha) = k^2 \int_0^\infty ds e^{-(\tilde{\omega}_\alpha - iv_\alpha k)s} \int_{-\infty}^\infty dq \hat{S}_\beta(q, s) \hat{S}_\gamma(k - q, s). \quad (23)$$

Using the scaling ansatz (12) we arrive at

$$I_{\beta\gamma}(k, \tilde{\omega}_\alpha) = k^2 \int_0^\infty ds e^{-(\tilde{\omega}_\alpha + i(v_\gamma - v_\alpha)k)s} A_{\beta\gamma}(k, s) \quad (24)$$

$$= k^2 \int_0^\infty ds e^{-(\tilde{\omega}_\alpha + i(v_\beta - v_\alpha)k)s} A_{\gamma\beta}(k, s) \quad (25)$$

with

$$A_{\beta\gamma}(k, s) = \int_{-\infty}^\infty dq e^{i(v_\gamma - v_\beta)qs} \hat{f}_\beta(qs^{\frac{1}{z_\beta}}) \hat{f}_\gamma((k - q)s^{\frac{1}{z_\gamma}}). \quad (26)$$

As pointed out above, in the static case $k = 0$ the constant solution to the mode coupling equations is exact. Therefore we can focus on the non-static case $k \neq 0$. With $k = |k| \text{sgn}(k)$ and the substitution of integration variables $|k|s \rightarrow s$ we obtain

$$I_{\beta\gamma}(k, \tilde{\omega}_\alpha) = |k| \int_0^\infty ds e^{-(\tilde{\omega}_\alpha |k|^{-1} + i(v_\gamma - v_\alpha) \text{sgn}(k))s} B_{\beta\gamma}(k, s) \quad (27)$$

$$= |k| \int_0^\infty ds e^{-(\tilde{\omega}_\alpha |k|^{-1} + i(v_\beta - v_\alpha) \text{sgn}(k))s} B_{\gamma\beta}(k, s) \quad (28)$$

with

$$B_{\beta\gamma}(k, s) = \int_{-\infty}^\infty dq e^{i(v_\gamma - v_\beta)q|k|^{-1}s} \hat{f}_\beta(q|k|^{-\frac{1}{z_\beta}} s^{\frac{1}{z_\beta}}) \hat{f}_\gamma((k - q)|k|^{-\frac{1}{z_\gamma}} s^{\frac{1}{z_\gamma}}). \quad (29)$$

Condition 2: Local interactions ($z_\alpha > 1 \forall \alpha$).

As discussed above, for sufficiently fast decaying interaction strength one expects that all modes spread sub-ballistically around their centers at $x_\alpha(t)$, i.e., $z_\alpha > 1 \forall \alpha$. Then the small- k behaviour of the integral (29) simplifies since the term $k|k|^{-\frac{1}{z_\gamma}}$ in the second argument vanishes. One is left with

$$B_{\beta\gamma}(k, s) = \int_{-\infty}^{\infty} dq e^{i(v_\gamma - v_\beta)q|k|^{-1}s} \hat{f}_\beta(q|k|^{-\frac{1}{z_\beta}} s^{\frac{1}{z_\beta}}) \hat{f}_\gamma(-q|k|^{-\frac{1}{z_\gamma}} s^{\frac{1}{z_\gamma}}). \quad (30)$$

For $v_\gamma = v_\beta$ this expression reduces to

$$B_{\beta\gamma}(k, s) = \int_{-\infty}^{\infty} dq \hat{f}_\beta(q|k|^{-\frac{1}{z_\beta}} s^{\frac{1}{z_\beta}}) \hat{f}_\gamma(-q|k|^{-\frac{1}{z_\gamma}} s^{\frac{1}{z_\gamma}}). \quad (31)$$

Taking $\beta = \gamma$ this yields the diagonal elements

$$\begin{aligned} B_{\beta\beta}(k, s) &= \int_{-\infty}^{\infty} dq \hat{f}_\beta(q|k|^{-\frac{1}{z_\beta}} s^{\frac{1}{z_\beta}}) \hat{f}_\beta(-q|k|^{-\frac{1}{z_\beta}} s^{\frac{1}{z_\beta}}) \\ &= |k|^{\frac{1}{z_\beta}} s^{-\frac{1}{z_\beta}} \Omega[\hat{f}_\beta] \end{aligned} \quad (32)$$

with the functional

$$\Omega[f] = \int_{-\infty}^{\infty} dk \hat{f}(k) \hat{f}(-k) = \int_{-\infty}^{\infty} dx (f(x))^2. \quad (33)$$

Thus we find from (27)

$$I_{\beta\beta}(k, \tilde{\omega}_\alpha) = |k|^{1+\frac{1}{z_\beta}} \Omega[\hat{f}_\beta] \int_0^\infty ds s^{-\frac{1}{z_\beta}} e^{-(\tilde{\omega}_\alpha |k|^{-1} + i(v_\beta - v_\alpha) \text{sgn}(k))s}. \quad (34)$$

With the scaling variable

$$\zeta_\alpha = \tilde{\omega}_\alpha |k|^{-z_\alpha} \quad (35)$$

and the shorthand

$$v_k^{\alpha\beta} = (v_\alpha - v_\beta) \text{sgn}(k) \quad (36)$$

this reads

$$I_{\beta\beta}(k, \zeta_\alpha) = |k|^{1+\frac{1}{z_\beta}} \Omega[\hat{f}_\beta] \int_0^\infty ds e^{-(\zeta_\alpha |k|^{z_\alpha-1} - i v_k^{\alpha\beta})s} s^{-\frac{1}{z_\beta}} \quad (37)$$

$$= |k|^{1+\frac{1}{z_\beta}} \Omega[\hat{f}_\beta] \Gamma\left(1 - \frac{1}{z_\beta}\right) \left(\zeta_\alpha |k|^{z_\alpha-1} - i v_k^{\alpha\beta}\right)^{\frac{1}{z_\beta}-1} \quad (38)$$

which also holds for $\beta = \alpha$. Here we have used the integral representation

$$\Gamma(x) = p^x \int_0^\infty du u^{x-1} e^{-pu} = p^x / x \int_0^\infty du e^{-pu^{1/x}} \quad (39)$$

for $\Re(x) > 0$, $\Re(p) > 0$ of the Gamma-function.

Condition 3: Strict hyperbolicity ($v_\beta \neq v_\gamma \forall \beta \neq \gamma$).

Up to this point the assumption of strict hyperbolicity has only led us to consider the mode-coupling equations in the form (13), but it has not yet entered their analysis. Strict hyperbolicity plays a role only in (30). We make the substitution of integration variables $q(s/|k|)^x \rightarrow q$ where $x = \max[\frac{1}{z_\beta}, \frac{1}{z_\gamma}] < 1$. Then (30) becomes

$$B_{\beta\gamma}(k, s) = |k/s|^x \int_{-\infty}^\infty dq e^{i(v_\gamma - v_\beta)q|k/s|^{x-1}} \hat{f}_\beta(q|k/s|^{x-\frac{1}{z_\beta}}) \hat{f}_\gamma(-q|k/s|^{x-\frac{1}{z_\gamma}}). \quad (40)$$

This leads to a term $|k|^{x-1} \rightarrow \infty$ in the exponential. Thus for $v_\gamma \neq v_\beta$ we have a rapidly oscillating term and the integral vanishes exponentially fast.

This proves that the leading contributions to the dynamical structure function come from the diagonal elements $\beta = \gamma$ of the mode coupling matrix. Therefore (22) reads

$$\tilde{S}_\alpha(k, \zeta_\alpha) = \frac{1}{\sqrt{2\pi}} |k|^{-z_\alpha} h_\alpha(\zeta_\alpha) \quad (41)$$

where from (38) we have

$$h_\alpha(\zeta_\alpha) = \lim_{k \rightarrow 0} \left[\zeta_\alpha + D_\alpha |k|^{2-z_\alpha} + Q_{\alpha\alpha} \zeta_\alpha^{\frac{1}{z_\alpha}-1} |k|^{3-2z_\alpha} + \sum_{\beta \neq \alpha} Q_{\alpha\beta} \left(\zeta_\alpha |k|^{z_\alpha-1} - i v_k^{\alpha\beta} \right)^{\frac{1}{z_\beta}-1} |k|^{1+\frac{1}{z_\beta}-z_\alpha} \right]^{-1}. \quad (42)$$

with the generally positive constants

$$Q_{\alpha\beta} = 2(G_{\beta\beta}^\alpha)^2 \Gamma\left(1 - \frac{1}{z_\beta}\right) \Omega[\hat{f}_\beta] \geq 0. \quad (43)$$

We invoke again strict hyperbolicity and subballistic scaling to deduce that the term $\zeta_\alpha |k|^{z_\alpha-1}$ in (42) can be neglected for the long wave length behaviour. This yields for the diagonal terms

$$h(\zeta_\alpha) = \lim_{k \rightarrow 0} \left[\zeta_\alpha + D_\alpha |k|^{2-z_\alpha} + Q_{\alpha\alpha} \zeta_\alpha^{\frac{1}{z_\alpha}-1} |k|^{3-2z_\alpha} \right]$$

$$+ \sum_{\beta \neq \alpha} Q_{\alpha\beta} \left(-iv_k^{\alpha\beta} \right)^{\frac{1}{z_\beta}-1} |k|^{1+\frac{1}{z_\beta}-z_\alpha} \Big]^{-1}. \quad (44)$$

This is the starting point for the subsequent analysis of the small- k behaviour. We remark that with the shorthand

$$\sigma_k^{\alpha\beta} = \text{sgn}[k(v_\alpha - v_\beta)] \quad (45)$$

we have

$$\left(-iv_k^{\alpha\beta} \right)^{\frac{1}{z_\beta}-1} = |v_\alpha - v_\beta|^{\frac{1}{z_\beta}-1} \exp \left(i\sigma_k^{\alpha\beta} \left(1 - \frac{1}{z_\beta} \right) \frac{\pi}{2} \right) \quad (46)$$

$$= \frac{\cos \left(\left(1 - \frac{1}{z_\beta} \right) \frac{\pi}{2} \right)}{|v_\alpha - v_\beta|^{1-\frac{1}{z_\beta}}} \left[1 + i\sigma_k^{\alpha\beta} \tan \left(\left(1 - \frac{1}{z_\beta} \right) \frac{\pi}{2} \right) \right] \quad (47)$$

$$= \frac{\sin \left(\frac{\pi}{2z_\beta} \right)}{|v_\alpha - v_\beta|^{1-\frac{1}{z_\beta}}} \left[1 - i\sigma_k^{\alpha\beta} \tan \left(\left(1 + \frac{1}{z_\beta} \right) \frac{\pi}{2} \right) \right] \quad (48)$$

In the last line we made use of $\tan(-x) = -\tan(x)$ and $\tan(x) = \tan(x - \pi)$.

C. Asymptotic analysis

Now one has to search for the dynamical exponents for which the limit $k \rightarrow 0$ is non-trivial, i.e., $h(\zeta_\alpha)$ finite and $h(\zeta_\alpha) \neq \zeta_\alpha$ (which would correspond to the δ -peak of the linear theory which does not exhibit the fluctuations). This has to be done self-consistently for all modes. Different self-consistency conditions arise depending on which diagonal elements of the mode-coupling matrices vanish. In the following we consider some fixed mode α and study all possible scenarios which depend on which is the smallest power in k in (44) that yields a non-trivial scaling form. To this end we define the set

$$\mathbb{I}_\alpha := \{\beta : G_{\beta\beta}^\alpha \neq 0\} \quad (49)$$

of non-zero diagonal mode coupling coefficients. Thus \mathbb{I}_α is the set of modes β that give rise to a non-linear term in the time-evolution of the mode α that one considers.

Case A: $\mathbb{I}_\alpha = \emptyset$

If mode α decouples, i.e., if *all* diagonal terms $G_{\beta\beta}^\alpha = 0$ then one has $h(\zeta_\alpha) = [\zeta_\alpha + D_\alpha |k|^{2-z_\alpha}]^{-1}$ and therefore

$$z_\alpha = 2 \quad (50)$$

and

$$\hat{S}_\alpha(k, t) = \frac{1}{\sqrt{2\pi}} e^{-iv_\alpha kt - D_\alpha k^2 t} \quad (51)$$

which is pure diffusion. (We remind the reader that we ignore possible logarithmic corrections from cubic contributions to the NLFH equations.)

From (51) we read off the scaling function

$$\hat{f}_\alpha(\kappa_\alpha) = \frac{1}{\sqrt{2\pi}} e^{-D_\alpha \kappa_\alpha^2} \quad (52)$$

with scaling variable $\kappa_\alpha = kt^{1/2}$. This yields

$$\Omega[\hat{f}_\alpha] = \frac{1}{2\sqrt{2\pi D_\alpha}} \quad \text{for diffusive modes } \alpha \quad (53)$$

and

$$Q_{\beta\alpha} = \frac{(G_{\alpha\alpha}^\beta)^2}{\sqrt{2D_\alpha}} \quad \text{for non-diffusive modes } \beta \neq \alpha. \quad (54)$$

Case B: $\alpha \notin \mathbb{I}_\alpha, \mathbb{I}_\alpha \neq \emptyset$

If $G_{\alpha\alpha}^\alpha = 0$, but some $G_{\beta\beta}^\alpha \neq 0$, then mode α has quadratic contributions from one or more other modes β . One has

$$h(\zeta_\alpha) = \lim_{k \rightarrow 0} \left[\zeta_\alpha + D_\alpha |k|^{2-z_\alpha} + \sum_{\beta \neq \alpha} Q_{\alpha\beta} \left(-iv_k^{\alpha\beta} \right)^{\frac{1}{z_\beta}-1} |k|^{1+\frac{1}{z_\beta}-z_\alpha} \right]^{-1}. \quad (55)$$

corresponding to

$$\hat{S}_\alpha(k, t) = \frac{1}{\sqrt{2\pi}} \exp \left(-iv_\alpha kt - \left[D_\alpha k^2 + \sum_{\beta} Q_{\alpha\beta} \left(-iv_k^{\alpha\beta} \right)^{\frac{1}{z_\beta}-1} |k|^{1+\frac{1}{z_\beta}} \right] t \right) \quad (56)$$

Since by Condition 2 one has $1 + \frac{1}{z_\beta} < 2$ it follows that $2 - z_\alpha > 1 + \frac{1}{z_\beta} - z_\alpha$. Hence the diffusive term in (56) is subleading and the dominant terms in (56) are those terms proportional to $(G_{\beta\beta}^\alpha)^2$ which have the largest z_β . We shall denote this value by z_β^{max} and define the set $\mathbb{I}_\alpha^* = \{\beta \in \mathbb{I}_\alpha : z_\beta = z_\beta^{max}\}$. This leads to

$$z_\alpha = \min_{\beta \in \mathbb{I}_\alpha} \left[\left(1 + \frac{1}{z_\beta} \right) \right] = 1 + \frac{1}{z_\beta^{max}} > 1. \quad (57)$$

Hence the assumption of subballistic scaling that arises from Condition 2 is self-consistent. The dynamical structure (56) reduces to

$$\hat{S}_\alpha(k, t) = \frac{1}{\sqrt{2\pi}} \exp \left(-iv_\alpha kt - \sum_{\beta \in \mathbb{I}_\alpha^*} Q_{\alpha\beta} \left(-iv_k^{\alpha\beta} \right)^{z_\alpha-2} |k|^{z_\alpha} t \right) \quad (58)$$

where from (48) and (57) we have

$$\left(-iv_k^{\alpha\beta} \right)^{z_\alpha-2} = \frac{\sin \left((z_\alpha - 1) \frac{\pi}{2} \right)}{|v_\alpha - v_\beta|^{2-z_\alpha}} \left(1 - i\sigma_k^{\alpha\beta} \tan \left(\frac{\pi z_\alpha}{2} \right) \right). \quad (59)$$

Defining

$$E_\alpha = \sum_{\beta \in \mathbb{I}_\alpha^*} Q_{\alpha\beta} \frac{\sin \left((z_\alpha - 1) \frac{\pi}{2} \right)}{|v_\alpha - v_\beta|^{2-z_\alpha}} \quad (60)$$

$$F_\alpha = \sum_{\beta \in \mathbb{I}_\alpha^*} Q_{\alpha\beta} \frac{\sin \left((z_\alpha - 1) \frac{\pi}{2} \right)}{|v_\alpha - v_\beta|^{2-z_\alpha}} \text{sgn}(v_\alpha - v_\beta) \quad (61)$$

$$A_\alpha = \frac{F_\alpha}{E_\alpha} \quad (62)$$

allows us to write

$$\hat{S}_\alpha(k, t) = \frac{1}{\sqrt{2\pi}} \exp \left(-iv_\alpha kt - E_\alpha |k|^{z_\alpha} t \left[1 - iA_\alpha \tan \left(\frac{\pi z_\alpha}{2} \right) \text{sgn}(k) \right] \right). \quad (63)$$

One recognizes in (63) an asymmetric α -stable Lévy-distribution with asymmetry $A_\alpha \in [-1, 1]$. If mode α is to the left or right of *all* modes with z_β^{max} that control it (i.e. if $v_\alpha < v_\beta \forall \beta \in \mathbb{I}_\alpha^*$ or if $v_\alpha > v_\beta \forall \beta \in \mathbb{I}_\alpha^*$, then $\sigma_k^{\alpha\beta}$ has the same sign for all $\beta \in \mathbb{I}_\alpha^*$ and as a consequence $A_\alpha = \pm 1$. This means that then the asymmetry is maximal. This is the classical analogue of the Lieb-Robinson bound which is a theoretical upper limit on the speed at which information can propagate in non-relativistic quantum systems [24].

From (63) we obtain the scaling function

$$\hat{f}_\alpha(\kappa) = \frac{1}{\sqrt{2\pi}} \exp \left(-E_\alpha |\kappa|^{z_\alpha} \left[1 - iA_\alpha \text{sgn}(\kappa) \tan \left(\frac{\pi z_\alpha}{2} \right) \right] \right) \quad (64)$$

which gives (see (33) and (39))

$$\Omega[\hat{f}_\alpha] = \frac{1}{\pi z_\alpha} (2E_\alpha)^{-\frac{1}{z_\alpha}} \Gamma \left(\frac{1}{z_\alpha} \right) \quad \text{for Fibonacci modes } \alpha. \quad (65)$$

Using the identity

$$\Gamma \left(1 - \frac{1}{x} \right) = \frac{\pi}{\Gamma \left(\frac{1}{x} \right) \sin \left(\frac{\pi}{x} \right)} \quad (66)$$

one finds

$$Q_{\alpha\beta} = \frac{2(G_{\beta\beta}^\alpha)^2 (2E_\beta)^{-\frac{1}{z_\beta}}}{z_\beta \sin\left(\frac{\pi}{z_\beta}\right)} \quad \text{for Fibonacci modes } \beta \neq \alpha \quad (67)$$

for the constant (43). We recall that E_α is not a simple constant depending only on mode α , but a functional that depends on all modes $\beta \in \mathbb{I}_\alpha^*$.

The upshot of cases A and B is that if $G_{\alpha\alpha}^\alpha = 0$ one has the bounds

$$1 < z_\alpha \leq 2 \quad (68)$$

for the dynamical exponents of modes whose self-coupling constant vanishes. The equality $z = 2$ is attained if and only if all diagonal coupling constants of that mode vanish. The relation (57) determines the dynamical exponents. The scaling functions are asymmetric Lévy functions.

Case C: $\alpha \in \mathbb{I}_\alpha$

For $G_{\alpha\alpha}^\alpha \neq 0$, i.e., non-vanishing quadratic self-coupling, imagine first that $z_{\beta^*} > 2$ for some mode β^* . Then according to (44) a non-trivial scaling form is obtained for the following values of the dynamical exponent: $z_\alpha = 1 + 1/z_{\beta^*} < 3/2$ (from the term proportional to $G_{\beta^*\beta^*}^\alpha$), $z_\alpha = 3/2$ (from the self-coupling term $G_{\alpha\alpha}^\alpha$), or $z_\alpha = 2$ (from the diffusive term). This excludes the possibility $z_\alpha > 2$ for $G_{\alpha\alpha}^\alpha \neq 0$. Above it was established that $z_\alpha \leq 2$ for $G_{\alpha\alpha}^\alpha = 0$. Thus we conclude that for *all* modes the bounds (68) are valid self-consistently. Therefore below we can assume without loss of generality $1 < z_\beta \leq 2$.

Next we observe the leading small- k behaviour of (44) with non-trivial scaling form is obtained for $z_\alpha = \min\{2, 3/2, 1 + 1/z_\beta\}$. Thus

$$z_\alpha = 3/2 \quad (69)$$

because of (68).

Even though the dynamical exponent is uniquely given by $z_\alpha = 3/2$ if $G_{\alpha\alpha}^\alpha \neq 0$, there are two different families of scaling functions. If $z_\beta < 2$ for all modes, i.e., if all modes have at least one non-zero diagonal element, then

$$h(\zeta_\alpha) = \left[\zeta_\alpha + Q_{\alpha\alpha} \zeta_\alpha^{-\frac{1}{3}} \right]^{-1}. \quad (70)$$

This corresponds to the usual KPZ-mode where mode-coupling theory is known to be quantitative quite good but not exact [25, 26]. On the other hand, if $z_\beta = 2$ for some diffusive modes from a set B^{diff} , then

$$h(\zeta_\alpha) = \left[\zeta_\alpha + Q_{\alpha\alpha}\zeta_\alpha^{-\frac{1}{3}} + \sum_{\beta \in B^{diff}} Q_{\alpha\beta} \left(-iv_k^{\alpha\beta} \right)^{-\frac{1}{2}} \right]^{-1}. \quad (71)$$

This corresponds to a modified KPZ-mode [3] which has not been studied yet in detail.

The constants defined in (43) are

$$Q_{\alpha\beta} = 2(G_{\beta\beta}^\alpha)^2 \Gamma(1/3) \Omega[\hat{f}_\beta] \text{ for } \beta = \text{KPZ, KPZ}' \quad (72)$$

for a KPZ or modified KPZ mode β . In order to compute $\Omega_{\text{KPZ}} \equiv \Omega[\hat{f}_{\text{KPZ}}]$ for $\beta = \text{KPZ}$ we use the exact scaling form $S_{\text{KPZ}}(x, t) = (\lambda t)^{-2/3} f_{\text{KPZ}}((x - v_\beta t)/(\lambda t)^{2/3})$ with $\lambda = 2\sqrt{2}|G_{\beta\beta}^\beta|$ [5]. With the scaling variable $\xi = (x - v_\beta t)/t^{2/3}$ as defined in (10) we obtain the real-space scaling function $f_\beta(\xi) = \lambda^{-2/3} f_{\text{KPZ}}(\lambda^{-2/3}\xi)$. Therefore, by definition we have

$$\Omega_{\text{KPZ}} = \int_{-\infty}^{\infty} d\xi (f_\beta(\xi))^2 = \lambda^{-2/3} \int_{-\infty}^{\infty} dx (f_{\text{KPZ}}(x))^2 = \frac{1}{2} (G_{\beta\beta}^\beta)^{-2/3} c_{PS}. \quad (73)$$

For the universal constant

$$c_{PS} := \int_{-\infty}^{\infty} dx (f_{\text{KPZ}}(x))^2 = 0.3898135914137278 \quad (74)$$

we do not have an expression in closed form but its value can be computed numerically with high precision from the Prähofer-Spohn scaling function $f_{\text{KPZ}}(x)$ tabulated in [29]. The double precision result (sixteen significant digits) shown in (74) is numerically exact and was obtained from the data in [29] by trapezoidal integration.[30] The scale factors E_α (60) that enter the scaling functions of Fibonacci modes with non-zero coupling to a KPZ mode are sensitive to c_{PS} and therefore a precise value is important for numerical fits. For the modified KPZ mode $\beta = \text{KPZ}'$ the functional $\Omega_{\text{KPZ}'}$ has the same form as (73), but the numerical value of the integral is not known since the scaling function $f_{\text{KPZ}'}(x)$ for the modified KPZ mode is not known.

D. Classification of universality classes

We set out to classify the possible universality classes. We summarize the equations that determine the dynamical exponents for a system with n modes:

$$z_\alpha = \begin{cases} 2 & \text{if } \mathbb{I}_\alpha = \emptyset \\ 3/2 & \text{if } \alpha \in \mathbb{I}_\alpha \\ \min_{\beta \in \mathbb{I}_\alpha} \left[\left(1 + \frac{1}{z_\beta} \right) \right] & \text{else} \end{cases} \quad (75)$$

and

$$1 < z_\alpha \leq 2 \quad \forall \alpha \quad (76)$$

In order to solve the non-linear recursion (75) in case B we iterate the recursion to find e.g. for a five-fold iteration the continued fraction

$$z_5 = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{z_1}}}}. \quad (77)$$

Here the modes are ordered in such a fashion that the mode that minimizes the exponent of mode 2 is mode 1 and so on. The continued fraction terminates when a set \mathbb{I}_β in this iteration of (75) is empty. Remarkably, for $z_1 = 1$ this is the well-known continued-fraction representation of the Kepler ratios which implies that if z_1 is any Kepler ratio F_i/F_{i+1} then $z_n = F_{n+i-1}/F_{n+i}$ is also a Kepler ratio. Thus for each parent critical exponent 2 or 3/2 from case A or case C (which are both Kepler ratios) one generates descendant dynamical exponents which are Kepler ratios as long as the sets \mathbb{I}_{β_i} are non-empty. If the lowest set \mathbb{I}_{β_1} is empty, i.e., if there is no coupling from any mode in case B to a mode from case A or C then the unique solution to the recursion is the golden mean $z_\alpha = \varphi$ for all modes from case B. The golden mean is defined by

$$\varphi := \frac{1}{2}(\sqrt{5} + 1). \quad (78)$$

Useful relations are

$$\varphi^{-1} = \frac{1}{2}(\sqrt{5} - 1), \quad \varphi = 1 + \varphi^{-1}, \quad \varphi^2 = 1 + \varphi, \quad \varphi^{-2} = 2 - \varphi. \quad (79)$$

The numerical value is $\varphi \approx 1.618$.

III. EXAMPLES

Given as input parameters the diagonal mode-coupling constants $G_{\beta\beta}^\alpha$, the diffusion coefficients D_α and the KPZ-functionals $\Omega[\hat{f}_{KPZ}]$, $\Omega[\hat{f}_{KPZ'}]$, the explicit scaling solutions of the mode-coupling equations are (51), (63), (70) and (71). The dynamical exponents z_α have to be determined self-consistently from the sets \mathbb{I}_α defined in (49), using (50), (57), (68) and (69). The prefactors of the scaling variable E_α for the Fibonacci modes are then given by (54), (60), (67), (72). The asymmetry for the Fibonacci modes is determined by (62). We stress that no assumptions other than strict hyperbolicity and subballistic scaling have been made to arrive at these results.

A. Example 1: $G_{11}^1 = G_{22}^1 = G_{22}^2 = 0$, $G_{11}^2 \neq 0$

Mode 1:

For mode 1 we have case A. Eq. (50) gives

$$z_1 = 2 \tag{80}$$

and (51) gives

$$\hat{S}_1(k, t) = \frac{1}{\sqrt{2\pi}} e^{-iv_1 kt - D_1 k^2 t} \tag{81}$$

which is diffusion.

Mode 2:

For mode 2 we have case B. Since there is only one other mode, which has $z_1 = 2$, (57) gives

$$z_2 = 3/2. \tag{82}$$

From (54) we obtain

$$Q_{21} = \frac{(G_{11}^2)^2}{\sqrt{2D_1}}, \tag{83}$$

from (60)

$$E_2 = Q_{21} \frac{\cos\left((2 - z_2) \frac{\pi}{2}\right)}{|v_2 - v_1|^{2-z_2}} = \frac{(G_{11}^2)^2}{2\sqrt{D_1}|v_2 - v_1|} \tag{84}$$

and from (62)

$$A_2 = \text{sgn}(v_2 - v_1). \quad (85)$$

Since $\tan(z_2\pi/2) = -1$ we arrive at

$$\hat{S}_2(k, t) = \frac{1}{\sqrt{2\pi}} \exp(-iv_2 kt - E_2|k|^{3/2}t(1 + i\text{sgn}(k(v_2 - v_1)))) \quad (86)$$

which is in agreement with [2].

B. Example 2: $G_{11}^1 = G_{22}^2 = 0$, $G_{22}^1, G_{11}^2 \neq 0$

For both modes we have case B. It is expedient to define

$$H_\alpha := 2E_\alpha \quad (87)$$

$$g_1 := (G_{22}^1)^2 \quad (88)$$

$$g_2 := (G_{11}^2)^2 \quad (89)$$

$$\theta := \frac{4 \sin((1 - \varphi)\frac{\pi}{2})}{\varphi \sin\left(\frac{\pi}{\varphi}\right) |v_1 - v_2|^{2-\varphi}} \quad (90)$$

For modes 1 and 2 we have from (57)

$$z_1 = 1 + \frac{1}{z_2}, \quad z_2 = 1 + \frac{1}{z_1} \quad (91)$$

The solution of these two equations is

$$z_1 = z_2 = \frac{1}{2}(1 + \sqrt{5}) = \varphi \quad (92)$$

(see also the relations (79)).

From (60) and (67) we obtain

$$H_1 = Q_{12} \frac{2 \sin((1 - \varphi)\frac{\pi}{2})}{|v_1 - v_2|^{2-\varphi}}, \quad H_2 = Q_{21} \frac{2 \sin((1 - \varphi)\frac{\pi}{2})}{|v_2 - v_1|^{2-\varphi}} \quad (93)$$

$$Q_{12} = \frac{2g_1 H_2^{-1/\varphi}}{\varphi \sin\left(\frac{\pi}{\varphi}\right)}, \quad Q_{21} = \frac{2g_2 H_1^{-1/\varphi}}{\varphi \sin\left(\frac{\pi}{\varphi}\right)}. \quad (94)$$

This yields

$$H_1 = \theta g_1 H_2^{-1/\varphi}, \quad H_2 = \theta g_2 H_1^{-1/\varphi}. \quad (95)$$

Solving for H_1 gives

$$H_1^{\varphi - \frac{1}{\varphi}} = \frac{(\theta g_1)^\varphi}{\theta g_2}. \quad (96)$$

Using the property $\varphi - 1/\varphi = 1$ of the golden mean we find

$$E_1 = \frac{1}{2} (\theta^2 g_1 g_2)^{\frac{\varphi-1}{2}} \left(\frac{g_1}{g_2} \right)^{\frac{\varphi+1}{2}} \quad (97)$$

Next we use the property of the golden mean to obtain

$$\frac{2 \sin((1-\varphi)\frac{\pi}{2})}{\sin(\frac{\pi}{\varphi})} = \frac{\sin((1-\varphi)\frac{\pi}{2})}{\sin(\frac{\pi}{2\varphi}) \cos(\frac{\pi}{2\varphi})} = \frac{1}{\sin(\frac{\pi\varphi}{2})}. \quad (98)$$

Thus

$$E_1 = \frac{1}{2} \left(\frac{2G_{22}^1 G_{11}^2}{\varphi \sin(\frac{\pi\varphi}{2}) |v_1 - v_2|^{2-\varphi}} \right)^{\varphi-1} \left(\frac{G_{22}^1}{G_{11}^2} \right)^{\varphi+1}. \quad (99)$$

With a similar calculation one obtains

$$E_2 = \frac{1}{2} \left(\frac{2G_{22}^1 G_{11}^2}{\varphi \sin(\frac{\pi\varphi}{2}) |v_1 - v_2|^{2-\varphi}} \right)^{\varphi-1} \left(\frac{G_{11}^2}{G_{22}^1} \right)^{\varphi+1}. \quad (100)$$

in agreement with [2] since $(2-\varphi)(1-\varphi) = 1 - 2/\varphi$.

Excursion: For $\lambda := G_{22}^1 = G_{11}^2$ and $c = -v_1 = v_2$ this case was treated in [3] in a different way. We demonstrate how the amplitude $E := E_1 = E_2$ arises from Eqs. (6.11), (6.12) and (6.14) in [3]. The point to prove is

$$E = C \quad (101)$$

where C is the amplitude of the scaling variable defined in the first line of (6.14).

Proof: We have to compute C from (6.11) and (6.12). To this end we define

$$\mu := 2(4\pi\lambda)^2 a, \quad \nu := \frac{1}{\gamma} \Gamma\left(\frac{1}{\gamma}\right) \quad (102)$$

with

$$a = (4\pi c)^{-1+1/\gamma} \frac{\pi}{2\Gamma\left(\frac{1}{\gamma}\right) \cos\left(\frac{\pi}{2\gamma}\right)} = (4\pi c)^{\gamma-2} \frac{\pi}{2\gamma\nu \sin\left(\frac{\pi\gamma}{2}\right)} \quad (103)$$

given in (6.11). In the second equality we used $\cos\left(\frac{\pi}{2\gamma}\right) = \sin\left(\frac{\pi\gamma}{2}\right)$ which follows from $1/\gamma = \gamma - 1$.

Now observe that (6.12) yields

$$A = (\mu A)^{-1/\gamma} \nu. \quad (104)$$

Taking this to the power γ and using $\gamma - 1 = 1/\gamma$ yields $A^\gamma = (\mu A)^{-1} \nu^\gamma = \mu^{-1+1/\gamma} \nu^{\gamma-1} A^{1/\gamma}$. Since $\gamma - 1/\gamma = 1$ we arrive at

$$A = \mu^{-1} (\mu \nu)^{1/\gamma} \quad (105)$$

where according to (103)

$$\mu \nu = 2(4\pi\lambda)^2 (4\pi c)^{\gamma-2} \frac{\pi}{2\gamma \sin(\frac{\pi\gamma}{2})} = \frac{4^\gamma \pi^{1+\gamma} \lambda^2}{\gamma \sin(\frac{\pi\gamma}{2})} c^{\gamma-2}. \quad (106)$$

This yields

$$(\mu \nu)^{1/\gamma} = 4\pi^\gamma \left(\frac{\lambda^2}{\gamma \sin(\frac{\pi\gamma}{2})} \right)^{1/\gamma} c^{1-2/\gamma}. \quad (107)$$

Now we note that by definition ((6.12) and first line of (6.14))

$$C = \frac{1}{2} \mu (2\pi)^{-\gamma} A \quad (108)$$

which gives

$$C = \frac{1}{4} 2^{1-\gamma} \pi^{-\gamma} (\mu \nu)^{1/\gamma} = \frac{1}{2^{1/\gamma}} \left(\frac{\lambda^2}{\gamma \sin(\frac{\pi\gamma}{2})} \right)^{1/\gamma} c^{1-2/\gamma} \quad (109)$$

Finally we rewrite E in terms of these parameters and $\varphi = \gamma$:

$$E = \frac{1}{2} \left(\frac{2\lambda^2}{\gamma \sin(\frac{\pi\gamma}{2})} \right)^{1/\gamma} (2c)^{1-2/\gamma} = \frac{1}{2^{1/\gamma}} \left(\frac{\lambda^2}{\gamma \sin(\frac{\pi\gamma}{2})} \right)^{1/\gamma} c^{1-2/\gamma} \quad (110)$$

which proves $C = E$. \square

C. Example 3: Two KPZ-modes and the heat mode

Consider three conservation laws and label the modes by 0 and $\sigma = \pm 1$. We consider $G_{\sigma\sigma}^\sigma = \gamma_s$, $G_{00}^0 = 0$ and $G_{11}^0 = -G_{-1-1}^0 = \gamma_h$. Furthermore we assume $v_\sigma = \sigma v$, $v_0 = 0$.

In this case $\mathbb{I}_\sigma^* = \{\sigma\}$ which means that the two modes $\sigma = \pm 1$ are KPZ. Following [1] they can be interpreted as sound modes and mode 0 is the heat mode. For the two sound modes one has [5]

$$\phi_\sigma(x, t) = (\lambda_s t)^{-2/3} f_{KPZ}((x - \sigma v t)/(\lambda_s t)^{2/3}) \quad (111)$$

with

$$\lambda_s = 2\sqrt{2}|\gamma_s| = 2^{3/2}|\gamma_s|. \quad (112)$$

Notice that $\lambda_s^{-2/3} = 1/2|\gamma_s|^{-2/3}$.

For the heat mode we find from (72) the constants $Q_{01} = Q_{0-1} = 2\gamma_h^2 \Gamma(1/3) \Omega_{KPZ}$. The structure of the mode-coupling matrices yields $\mathbb{I}_0^* = \{1, -1\}$. Therefore $z_0 = 5/3$ and from (60) one has $E_0 = 2Q_{01} \sin(\pi/3)v^{-1/3}$, $F_0 = 0$. Thus (62) gives $A_0 = 0$ and

$$\hat{S}_0(k, t) = \frac{1}{\sqrt{2\pi}} \exp(-E_0 |k|^{5/3} t) \quad (113)$$

with

$$E_0 = 2\Gamma\left(\frac{1}{3}\right) \sin\left(\frac{\pi}{3}\right) \gamma_h^2 v^{-1/3} \gamma_s^{-2/3} c_{PS}. \quad (114)$$

In order to see that this agrees with Eq. (4.12) of Ref. [5] one has to show that $E_0 = \lambda_h(2\pi)^{-5/3}$ with

$$\lambda_h = \lambda_s^{-2/3} (G_{\sigma\sigma}^0)^2 (4\pi)^2 (2\pi c)^{-1/3} \frac{\pi}{2\Gamma(2/3) \cos(\pi/3)} c_{PS} \quad (115)$$

and $v = c$. Indeed, one has, using (66) with $x = 3/2$,

$$\begin{aligned} \lambda_h(2\pi)^{-5/3} &= 4\lambda_s^{-2/3} (G_{\sigma\sigma}^0)^2 v^{-1/3} \frac{\pi}{2\Gamma(2/3) \cos(\pi/3)} c_{PS} \\ &= 4\gamma_h^2 v^{-1/3} \lambda_s^{-2/3} \frac{\Gamma(1/3) \sin(2\pi/3)}{2 \cos(\pi/3)} c_{PS} \\ &= 4\gamma_h^2 v^{-1/3} \lambda_s^{-2/3} \Gamma(1/3) \sin(\pi/3) c_{PS} \\ &= 2\gamma_h^2 v^{-1/3} \gamma_s^{-2/3} \Gamma(1/3) \sin(\pi/3) c_{PS} \\ &= E_0 \end{aligned} \quad (116)$$

which is what needed to be shown.

IV. CONCLUSIONS

We have shown that in the scaling limit the one-loop mode-coupling equations for the dynamical structure function for an arbitrary number of conservation laws in the strictly hyperbolic setting can be solved exactly. The solution yields a discrete family of dynamical universality classes with dynamical exponents that are the Kepler ratios $z_i = F_{i+2}/F_{i+1}$ which are in the range $3/2 \leq z_i \leq 2$. The largest exponent $z_1 = 2$ corresponds to a Gaussian diffusive mode, possibly with logarithmic corrections (that we did not consider). The smallest exponent $z_2 = 3/2$ represents three distinct universality classes with different scaling forms of the dynamical structure function: One has the KPZ universality class with the Prähofer-Spohn scaling function [27, 28], a modified KPZ universality class with

unknown scaling function [3], and a Fibonacci mode where the scaling function is given by the 3/2-Lévy stable distribution [2, 3, 15].

All higher modes $i \geq 3$ are Fibonacci modes with z_i -Lévy stable distributions as scaling functions, including the golden mean $z_\infty = \varphi$. In order to have a mode i with dynamical exponent z_i one needs at least $i - 1$ conservation laws, with the exception of the golden mean which requires only two conservation laws and always appears at least twice. Thus we have shown that diffusion, KPZ, modified KPZ and the Fibonacci family provide a *complete* classification of the dynamical universality classes which we expect to be generic for one-dimensional conservative systems where the long-time dynamics are dominated by the long-wave length behaviour of the modes associated with the conservation laws.

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 - [30] Using the data tabulated in [29] one can calculate $f_{\text{KPZ}}(x)$ with at least 90 digits accuracy in the interval $x \in [-8.5, 8.5]$. From this one can achieve with trapezoidal integration a much higher accuracy of c_{PS} than given here. Notice a small but significant numerical error of just over 10% in the value of c_{PS} given below Eq. (10) in Ref. [11].